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Study On Inverse Z-transform

Hawa Elhadi Eltaweel Manal Altaher Elzidani
h.eltaweel@edu.misuratau.edu.ly m.elzidani@edu.misuratau.edu.ly
Faculty of Education- Misurata University- Misurata- Libya

Abstract

To know the relationship between the Z-transform, Fourier transform and Laplace transform we need to know how to move from the Z-domain to the discrete time sequence in other words we need the inverse Z-transform.

In this paper we study methods for finding the inverse of the Z-transform as well as some important properties and examples of them. This paper illustrates the relation between Z-transform and Laplace transform and between Z-transform and Fourier transform Moreover we deduced the most important correspondence between the Z-plane and S- plane.

Keywords: Invers Z-transform, Laplace transform, Fourier transform, Region of convergence (ROC), Discrete time signal, continuous time signal.

دراسة على معكوس تحويل Z–

حواء الهادي الطويل منال الطاهر الزيداني

قسم الرياضيات– كلية التربية– جامعة مصراتة

الملخص بالعربي

لمعرفة العلاقة بين تحويل–Z وتحويل لابلاس نحتاج إلى الانتقال من نطاق –Z إلى التسلسل الزمني المتقطع وبمعنى اخر نحتاج إلى تحويل –Z المعكوس وفي هذا البحث تم دراسة طرق إيجاد معكوس –Z وذكر بعض الأمثلة على كل طريقة توضح دراسة خصائصها. وتوصلت الدراسة في هذا البحث إلى استنتاج العلاقة بين تحويل –Z وتحويل لابلاس وفورييه واستنتاج أهم نقاط التوافق بين هذه التحويلات.

الكلمات المفتاحية: معكوس تحويل –Z، تحويل لابلاس، تحويل فورييه، منطقة التقارب، الزمن المتقطع، الزمن المستمر.

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1. Introduction

Transformation is very powerful mathematical tool so using it in mathematical treatment of problem is arising in many application.
The idea of Z-transform back to 1730 when De Moivre introduced the concept of "generating function" to probability theory[Jury, 1964,8,9]
In1947 a transform of sampled signal or sequence defined by W. Hurewicez as a tractable way to solve linear constant-coefficients difference equations. The transformation named "Z-transform "by Ragazzini and Lotfi Zadeh in the sampled -data control group at Columbia University in 1952.
Z-transform is transformation for discrete data equivalent to the Laplace transform of continuous data and it's a generalization of discrete Fourier transform [Lawrenge, Ronald and Charles,1969,1949].
Recently there are many topic related to this topic [Behera and Rath ,2021,a^{572}] presented what Z-transform plays in the analysis of discrete signal in accordance with Laplace transform plays in the analysis of continuous signal also it represented the region of convergence and relationship of Z-transform and Laplace transform . [Juric . 2023,20] proposed a simple elegant and student-friendly way to calculate the inverse -transform of a rational function. In addition to the basic variant, which is most suitable for manual calculation when the poles are relative nice numbers. Moreover [Kallolkar and Salunke, 2015,16] made an attempt of study of Fourier Transform, Laplace Transform and Z-transform. It also showed sequential mathematical flow of interlinking of the three transform.
Some basic definition and concepts of sequences are presented on integration on complex plane.
Methods for determining the invers of Z-transform are represented, also we have discussed the relation between Z-transform and Laplace transform and discrete Fourier transform. At the end the most important correspondence between the S-plane and Z-plan are concluded.
Now we will give some basic definition, concepts and theorems important for our research.

Definition 1: [Poularikas,1999,91**]**
The complex sequence $\{a_k\}$ is called geometric sequence if ∃ a constant s ∈ □ s.t

$$\frac{a_{k+1}}{a_k} = s, \forall\, k \in \mathbb{N} \qquad (1.1)$$

In that case

$$a_k = as^k \qquad (1.2)$$

A geometric series is of the form

$$\sum_{k=0}^{\infty} as^k = a + as + as^2 + \cdots \qquad (1.3)$$

Note that a finite geometric series is summable with

$$\sum_{k=0}^{n} as^k = \begin{cases} \dfrac{a(1-s^{n+1})}{1-s}, s \neq 1 \\ a(n+1), s = 1 \end{cases} \qquad (1.4)$$

If $|s| < 1$,then

$$\sum_{k=0}^{\infty} as^k = \frac{a}{1-s} \qquad (1.5)$$

Definition 2: [Oppenheim, Willsky and Nawab,1992,8,9]
A sequence is called:
a) causal if:

$$x(n) = 0\,, for\;\; n < 0 \qquad (1.6)$$

b) anticausal if:

$$x(n) = 0\,, for\;\; n \geq 0 \qquad (1.7)$$

Definition 3: [Elattar,2007,6]
The unit step sequence or Heaviside step sequence is defined as

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad (1.8)$$

And for two-dimensional space it has the form

$$u(n,m) = \begin{cases} 1, & n,m \geq 0 \\ 0, & otherwie \end{cases} \qquad (1.9)$$

And the unit impulse or unit sample sequence is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \qquad (1.10)$$

And for two-dimensional space it has the form

$$\delta(n,m) = \begin{cases} 1, & n = m = 0 \\ 0, & otherwise \end{cases} \qquad (1.11)$$

Definition 4: [[Rabiner,Schafer and Rader,1969,1269]
The convolution between two infinite sequences $x(n)$
And $y(n)$ is defined as

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) \qquad (1.12)$$

And the correlation between two sequences $x(n)$ and $y(n)$ is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-1)$$
$$= x(l) * y(-l) \qquad (1.13)$$

Where l is an integer.

And the autocorrelation $r_{xx}(l)$ of a sequence $x(n)$ is the correlation with itself.

Before starting the methods of finding the invers -transform we need to these theorems.

Theorem 1: [churchil,2004,199] Laurent's Theorem

Suppose that a function $f(z)$ is analytic throughout an annular domain $r < |z - z_0| < R$, centered at $z_{0,}$ and let C be any positively oriented simple closed contour a round z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation.

$$f(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0), \qquad r < |z - z_0| < R \qquad (1.14)$$

Where

$$b_n = \frac{1}{2\pi j}\oint_c \frac{f(z)}{(z - z_0)} dz, \quad n \in \mathbb{Z} \qquad (1.15)$$

The representation of $f(z)$ in Eq(1.13) is called a Laurent series.

Definition 5 :[churchil,2004,231]

If z_0 is an isolated singular point of a function $f(z)$, then there is appositive number R such that $f(z)$ is analytic at each point z for which $0 < |z - z_0| < R$. Consequently, $f(z)$ has a Laurent series representation as in Eq(1.13). The complex number b_{-1} ,which is the coefficient of $1/(z - z_0)$ in Eq(1.13) is called the residue of at the isolated singular point z_0 , and we shall often write

$$b_{-1} = Res[f(z), z_0]$$

Theorem 2: [churchil,2004,244]

An isolated singular point z_0 of a function $f(z)$ is a pole of order s if and only if $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z-z_0)^s}$$

Where $g(z)$ is analytic and nonzero at z_0.
Moreover, if $s = 1$ then,

$$Res[f(z), z_0] = Res\left[\frac{g(z)}{z-z_0}, z_0\right] = g(z_0) \qquad (1.16)$$

And if $s \geq 2$

$$Res[f(z), z_0] = Res\left[\frac{g(z)}{(z-z_0)^s}, z_0\right]$$

$$= \frac{1}{(s-1)!}\frac{d^{s-1}}{dz^{s-1}} g(z)|z = z_0 \qquad (1.17)$$

Theorem 3: [churchil,2004,234] Cauchy's Residue Theorem.
If a function $f(z)$ is analytic inside and on a simple closed contour C (described in the positive sense) except for a finite number of singular points inside then,

$$\frac{1}{2\pi j}\oint_c f(z)dz = \sum_{k=1}^{n} Res[f(z), z_k] \qquad (1.18)$$

Theorem 4: [churchil,2004,189] Taylore's Theorem
Suppose that a function $f(z)$ is analytic throughout a disk $|z - z_0| < R$ centered at z_0 and with radius R. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, |z-z_0| < R \qquad (1.19)$$

Where

$$a_n = \frac{f^{(n)}(z)}{n!}, \quad n = 0,1,2, \dots \qquad (1.20)$$

The series in Eq(1.19) is called the Taylor series of $f(z)$ about $z = z_0$ and its converges to $f(z)$ when z lies in the given open disk.

Definition 6: [Elattar,2007,15]
Given an infinite complex sequence $x(n)$ we define its Z-transform $X(z)$ by the two sided infinite power series

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad (1.21)$$

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Where z is a complex variable, his Z-transform is called a two-sided or a bilateral Z -transform.

Note: Whenever we talk about Z-transform we mean the two-sided Z-transform.

Z-transform exists only for those values of Z for which the series in Eq(1.21) converges. These value of z define the region of convergence ROC of $X(z)$. Thus, the ROC is the domain of the Z-transform. So, whenever $X(z)$ is found it should be also defined.

The region of convergence of $X(z)$ is identical to the set of all values of Z which make the sequence in Eq(1.21) absolutely summable, i.e

$$\sum_{n=-\infty}^{\infty} |x(n)z^{-n}| < \infty \qquad (1.22)$$

Example 1

Determine the z-transform of the sequence

$$x(n) = \{\dots,0,1,\underset{\uparrow}{2},0,0,7,0,0,\dots\}$$
$$= \begin{cases} 1, & n=-2 \\ 2, & n=0 \\ 7, & n=3 \\ 0, & otherwise \end{cases}$$

Note: The arrow in this sequence indicates the position of $x(0)$.

Solution

$$\mathrm{X(z)} = \mathrm{Z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = z^2 + 2 + 7z^{-3}$$

$X(z)$ Converges for the entire z-plane except $z=0$ and $z=\infty$, so $Roc: 0 < |z| < \infty$

2. The inverse *z*-transform

Just as important as technique for finding the Z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence $x(n)$ from $X(z)$ Three methods are often used for the evaluation of the inverse of Z-transform.

1. Integration.
2. Power Series.
3. Partial-Fraction.

1.Integration Method

Integration method relies on Cauchy integral formula, which state that if C is a closed contour that encircles the origin in a counterclockwise direction then,

$$\frac{1}{2\pi j}\oint_c z^{k-1}dz = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad (2.1)$$

The Z-transform of a sequence is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad (2.2)$$

Multiply both sides of Eq(3.2)by $\frac{1}{2\pi j}z^{k-1}$ and integrating over a contour C that encloses the origin counterclockwise and lies entirely in the region of convergence of $X(z)$, we obtain

$$\frac{1}{2\pi j}\oint_c X(z)z^{k-1}dz = \frac{1}{2\pi j}\oint_c \sum_{n=-\infty}^{\infty} x(n)z^{-n+k-1}dz \qquad (2.3)$$

Interchanging the order of integration and summation on the right-hand side of Eq(2.3)(valid if the series is convergent) we get

$$\frac{1}{2\pi j}\oint_c X(z)z^{k-1}dz = \sum_{n=-\infty}^{\infty} x(n)\frac{1}{2\pi j}\oint_c z^{-n+k-1}dz \quad (2.4)$$

Applying Cauchy integral formula on the integral in the right hand side of Eq(2.4),we get

$$\frac{1}{2\pi j}\oint_c z^{-n+k-1}dz = f(x) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \qquad (2.5)$$

So Eq(2.4) becomes

$$\frac{1}{2\pi j}\oint_c X(z)z^{k-1}dz = x(k) \qquad (2.6)$$

Therefore, the inverse of z-transform is given by the integral

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz \qquad (2.7)$$

Where C is a counterclockwise closed contour in the region of convergence of $X(z)$ and encircling the origin of the z-plane and $n \in \mathbb{Z}$.

Note: For a rational z-transform is often evaluated using the residue theorem, i,e,

$$x(n) = \sum [residue\ of\ X(z)z^{n-1} at\ the\ poles\ inside\ C] \quad (2.8)$$

Example 2

Find the inverse z-transform of

$$X(z) = \frac{z}{z+3}, |z| > 3$$

Solution

From Eq(2.7),$x(n)$ will be

$$x(n) = \frac{1}{2\pi j} \oint_c \frac{z}{z+3} z^{n-1} dz = \frac{1}{2\pi j} \oint_c \frac{z^n}{z+3} dz$$

Where the contour of integration, C is a circle of radius greater than 3.

For $n \geq 0$, the contour of integration encloses only one pole at $z = -3$. So

$$x(n) = Res\left[\frac{z^n}{z+3}, -3\right] = (-3)^n$$

For $n < 0$ in an addition to the pole at $z = -3$ there is a multiple-order pole at $z = 0$ whose order depends on n

For $n = -1$

$$x(-1) = \frac{1}{2\pi j} \oint_c \frac{z}{z(z+3)} dz$$

$$= Res\left[\frac{1}{z(z+3)}, 0\right] + Res\left[\frac{1}{z(z+3)}, -3\right] = \frac{1}{3} + \frac{-1}{3} = 0$$

For $n = -2$

$$x(-2) = \frac{1}{2\pi j} \oint_c \frac{z}{z^2(z+3)} dz$$

$$= Res\left[\frac{1}{z^2(z+3)}, 0\right] + Res\left[\frac{1}{z^2(z+3)}, -3\right]$$

$$= \frac{1}{1!} \frac{d}{dz} \frac{1}{(z+3)} |z = 0 + \left[\frac{1}{9}\right] = \frac{-1}{9} + \frac{1}{9} = 0$$

Continuing this procedure it can be verified that $x(n) = 0, n < 0$. So

$$x(n) = \begin{cases} (-3)^n, & n \geq 0 \\ 0 \quad , & n < 0 \end{cases}$$

Or

$$x(n) = (-3)^n u(n)$$

Example 3

Find the inverse z-transform of

$$x(z) = z^2 + 6 + 7z^{-3}, 0 < |z| < \infty$$

Solution

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From Eq(2.7) $x(n)$will be

$$x(n) = \frac{1}{2\pi j}\oint_c (z^2 + 6 + 7z^{-3})z^{n-1}dz$$

.where is the unit circle taken counterclockwise.

$$x(n) = \frac{1}{2\pi j}\left[\oint_c z^{n+1}dz + 6\oint_c z^{n-1}dz + \oint_c z^{n-4}dz\right]$$

From Cauchy integral formula in we get,

$$x(-2) = \frac{1}{2\pi j}[2\pi j.1 + 0 + 0] = 1$$

$$x(0) = \frac{1}{2\pi j}[0 + 6\,.\; 2\pi j + 0] = 6$$

$$x(3) = \frac{1}{2\pi j}[0 + 0 + 7\,.2\pi j] = 7$$

For $n \neq -2,0,3$

$$x(n) = 0$$

So

$$\therefore x(n) = \begin{cases} 1, & n = -2 \\ 6, & n = 0 \\ 7, & n = 3 \\ 0, & otherwise \end{cases}$$

Note : Integration method is particularly useful if only a specific values of $x(n)$ are needed.

We now need to prove the Multiplication of two Sequence Property and Parseval,s Theorem.

Theorem 5: Multiplication of two sequences property

If $X(z)$ is the z-transform of $x(n)$ withROC $r_x < |z| < R_x$ and$Y(z)$ is the Z-transform of $y(n)$ with ROC $r_y < |z| < R_{y,}$ then

$$Z\big(x(n)y(n)\big) = \frac{1}{2\pi j}\oint_c X(v)Y\left(\frac{Z}{v}\right)dv,$$

$ROC\colon r_x r_y < |z| < R_x R_y$

Where is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(v)$and $Y\left(\frac{z}{v}\right)$

Proof

Let C be as given in the above theorem and

$$w(n) = x(n)y(n) \tag{2.9}$$

Then, the Z-transform of $w(n)$ is

$$w(z) = \sum_{n=-\infty}^{\infty} x(n)y(n)z^{-n} \tag{2.10}$$

But

$$x(n) = \frac{1}{2\pi j}\oint_c X(v)v^{n-1}\,dv \tag{2.11}$$

Substituting $x(n)$ in Eq(2.10) and interchanging the order of summation and integration we obtain

$$W(z) = \frac{1}{2\pi j}\oint_c X(v)\left[\sum_{n=-\infty}^{\infty} y(n)\left(\frac{z}{v}\right)^{-n}\right]\frac{1}{v}dv \tag{2.12}$$

The sum in brackets of Eq(2.12) is simply the transform evaluated at z/v Therefore,

$$w(z) = \frac{1}{2\pi j}\oint X(v)Y\left(\frac{z}{v}\right)\frac{1}{v}dv$$

To obtain the ROC of $W(z)$ we note that if $X(v)$ converges for $r_x < |v| < R_x$ and $Y(z)$ converges for $r_y < |z| < R_y$, then the ROC of $Y\left(\frac{z}{v}\right)$ is $r_y < \left|\frac{z}{v}\right| < R_y$

Hence the ROC for $W(z)$ is at least

$$r_x r_y < |z| < R_x R_y$$

Parseval's Theorem 6

Let $X(z)$ be the Z-transform of $x(n)$ with ROC $r_x < |z| < R_x$ and let $Y(z)$ be the z -transform of $y(n)$ with ROC $r_y < |z| < R_y$ with $r_x r_y < |z| = 1 < R_x R_y$

Then we have

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_c X(z)Y^*\left(\frac{1}{z^*}\right)\frac{1}{z}dz \tag{2.13}$$

Where$C$ is a counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(z)$ and $Y^*\left(\frac{1}{z^*}\right)$

Proof

Let

$$w(n) = x(n)y^*(n) \tag{2.14}$$

Noting that

$$\sum_{n=-\infty}^{\infty} w(n) = W(z)|z = 1 \tag{2.15}$$

By the multiplication of two sequences and the conjugation of Z-transform properties, Eq(2.15) becomes

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_c X(v)Y^*\left(\frac{1}{v^*}\right)\frac{1}{v}dv \qquad (2.16)$$

Replace the dummy variable v with z in Eq(2.16) we obtain

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint_c X(z)Y^*\left(\frac{1}{z^*}\right)\frac{1}{z}dz$$

Where C is counterclockwise closed contour that encloses the origin and lies within the common region of convergence of $X(z)$ and $Y^*(1/z^*)$

2. Power Series Method.

The idea of this method is to write $X(z)$ as a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \qquad (2.17)$$

Which convergence in the ROC. Then by uniqueness of $X(z)$ we can say that $x(n) = c_n$ for all n.

Example 4

Find the inverse z-transform of

$$X(z) = z^2 + 6 + 7z^{-3}, \qquad 0 < |z| < \infty$$

Solution

Since $X(z)$ is a finite -order integer power function, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-1} in $X(z)$.

Thus,$x(3) = 7, x(0) = 6$ and $x(-2) = 1$

$$\therefore x(n) = \begin{cases} 1, & n = -2 \\ 6, & n = 0 \\ 7, & n = 3 \\ 0, & otherwise \end{cases}$$

Note: comparing example 4 with example 3 we note the power series method is easier than integration method in such cases.

Example 5

Find the inverse -transform of

$$a)\ X(z) = \frac{z}{z+3}\ , |z| > 3$$

$$b)\ X(z) = \propto \frac{z}{z+3}\ , |z| < 3$$

Solution

a) Since the region of convergence is the exterior of a circle, the sequence is a right-sided sequence. Furthermore, since $\lim_{z\to\infty} X(z) = 1 =$ constant, it's a causal sequence. By long division to obtain power series in z^{-1}

$$\therefore X(z) = 1 - 3z^{-1} + 9z^{-2} + \cdots = \sum_{n=0}^{\infty} (-3)^n z^{-n}$$

So that

$$x(n) = (-3)^n u(n)$$

b) Since the region of convergence is the interior of a circle, the sequence is a left-sided sequence and since $X(0) = 0$ the sequence is anticausal. Thus we divide to obtain a series in power of z.

$$X(z) = \frac{1}{3}z - \frac{1}{9}z^2 + \frac{1}{27}z^3 + \cdots = \sum_{n=-\infty}^{\infty} -(-3)^n z^{-n}$$

$$\therefore x(n) = -(-3)^n u(-n-1)$$

3.Partial -Fraction Method.

If $X(z)$ is rational function then the partial fraction method is often useful method to find its inverse . The idea of this method is to write$X(z)$ as

$$X(z) = \propto_1 X_1(z) + \propto_2 X_2(z) + \cdots + \propto_{\boldsymbol{k}} X_k(z) \qquad (2.18)$$

Where each $X_i(z)$ has invers Z-transform $x_i(n)$ and $\propto_i \in \mathbb{C}$ for $i = 1,2, \dots, k$.

If $X(Z)$ is rational function of z then $X(z)$ can be expressed as

$$X(z) = \frac{A(z)}{B(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_M z^M}{b_0 + b_1 z + b_2 z^2 + \cdots + b_N z^N} \qquad (2.19)$$

Where $M, N \in \mathbb{N}$

If $N \neq 0$ and $M < N$ then $X(z)$ is called proper rational function, otherwise $X(z)$ is improper, and it can always be written as a sum of polynomial and proper rational function. i.e

$$X(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_k z^k + \frac{A(z)}{B(z)} \qquad (2.20)$$

The invers of the polynomial can be easily found but to find the inverse of the proper rational function we need to write it as a sum of simple function, for this purpose we factorized the denominator into factor of poles $p_0, p_1, \dots, p_m$ of $X(z)$.

Remark: Sometimes it's better to expand $X(z)/z$ rather than $X(z)$ because most Z-transform have the term z in their numerator.

We have two cases for the poles .

Case 1: Simple poles.

If all poles of $X(z)$ are simple then

$$X(Z) = \sum_{i=1}^{m} \frac{A_i}{z - p_i} \qquad (2.21)$$

Where

$$A_i = (z - p_i)X(z)|z = p_i \qquad (2.22)$$

Then

$$Z^{-1}\left[\frac{z}{z - p_i}\right] = \begin{cases} (p_i)^n u(n), & if\ ROC\ \ |z| > |p_i| \\ -(p_i)^n u(-n-1), & if\ ROC\ \ |z| < |p_i| \end{cases} \qquad (2.23)$$

If $x(n)$ is causal and some poles of $X(z)$ are complex then if p is a pole then p^* is also a pole and in this case

$$x(n) = [A(p)^n + A^*(p^*)^n]u(n) \qquad (2.24)$$

In polar form Eq(2.24) become

$$x(n) = 2|A||p|^n \cos(\theta n + \varphi)\, u(n) \qquad (2.25)$$

Where θ and φ are the argument of the pole P and the argument of the partial fraction coefficient A ,respectively.

Case 2: Multiple poles.

For a function $X(z)$ with a repeated pole of multiplicity r, r partial fraction coefficients are associated with this repeated pole. The partial fraction expansion of $X(z)$ will be of the form

$$X(Z) = \sum_{k=1}^{r} \frac{A_{1k}}{(z - p_1)^{r+1-k}} + \sum_{k=r+1}^{m} \frac{A_k}{z - p_k} \qquad (2.26)$$

Where

$$A_{1k} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z - p_1)^r X(z)|z = p_1 , k = 1,2, \dots, r \qquad (2.27)$$

Example 6

Find the inverse z-transform of

$$X(z) = \frac{z^2 + 3z}{z^2 - 3z + 2}$$

If ROC is

a) $|z| > 1$

b) $|z| < 2$

c) $1 < |z| < 2$

Solution

First we write $X(z)/z$ as a partial fraction
So
$$X(z) = 5\frac{z}{z-2} - 4\frac{z}{z-1}$$
For (a):
Since the ROC of $X(z)$ is $|z| > 1$ the sequence $x(n)$ is causal sequence so we obtain
$$x(n) = (5(2)^n - 4)u(n)$$
b) The ROC of $X(z)$ is $|z| < 2$ so the sequence $x(n)$ is anticausal so,
$$x(n) = (-5(2)^n + 4)u(-n-1)$$
c)The last ROC $1 < |z| < 2$ of $X(z)$ is annular, so the sequence $x(n)$ is two-sided. Thus one of the terms is causal and the other is anticausal. The ROC is overlapping $|z| > 1$ and $|z| < 2$ so the pole $p_1 = 1$ provides the causal sequence and the pole $p_2 = 2$ provides the anticausal sequence. Thus
$$x(n) = -4u(n) - 5(2)^n u(-n-1)$$
Example 7
Find the inverse z-transform of
$$X(z) = \frac{z+1}{z^2 - 2z + z}\ ROC\ |z| > \sqrt{2}$$
Solution
We write $Y(Z) = X(Z)/Z$ as a partial fraction
So,
$$x(n) = f(x) = \begin{cases} -\left(\frac{1+3j}{4}\right)(1+j)^n - \left(\frac{1-3j}{4}\right)(1-j)^n, & n \geq 1 \\ 0 \quad , & otherwise \end{cases}$$

For $n \geq 1, x(n)$ can be written in polar form as:
$$x(n) = \left(\frac{\sqrt{10}}{2}\right)\left(\sqrt{2}\right)^n \cos\left(\frac{\pi}{4}n + \tan^{-1}3\right)$$
$$= \sqrt{5}\left(\sqrt{2}\right)^{n-1} \cos\left(\frac{\pi}{4}n + \tan^{-1}3\right)$$
3.1The Relation Between Z-transform and the Discrete Fourier Transform.
Let $x(n)$ be sequence, then the discrete Fourier transform (DFT) of $x(n)$ is
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-n\omega j} \quad (3.1)$$

Where ω is real.
The Z-transform of x(n) is

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n} \qquad (3.2)$$

Where z is a complex variable.
In polar form z is written as

$$z = re^{j\omega} \qquad (3.3)$$

Where $r = |z|$ and $\omega = \arg(z)$.
Substituting Eq(3.3) in Eq(3.2) we obtain

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} x(n) r^{-n}\left(e^{j\omega}\right)^{-n} \quad (3.4)$$

Which is the discrete Fourier-transform for $[x(n)r^{-n}]$. If $r = 1$ then $z = e^{j\omega}$ and the Z-transform of $x(n)$ becomes the discrete Fourier-transform. i.e

$$X(z)\big|z = e^{j\omega} \; = X(e^{j\omega})$$

So, Z-transform is generalization of discrete Fourier-transform

3.2The Relation Between *Z*-transform and Laplace Transform

Let $f(t)$ be a continuous function, then we can take a sample discrete function $f_s(t)$ which can be written as

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\delta\,(n-t) = \sum_{n=-\infty}^{\infty} f(n)\delta(n-t) \qquad (3.5)$$

The Laplace transform of sampled function $f_s(t)$ is:

$$X(s) = \mathcal{L}[f_s(t)] = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} f(n)\delta(t-n)\right] e^{-st} dt \qquad (3.6)$$

Where $s = \sigma + j\omega, \sigma$ and ω are real variable, by interchanging the order of the summation and the integration of Eq(3.6) we get

$$X(s) = \sum_{n=-\infty}^{\infty} f(n)\left[\int_{-\infty}^{\infty} \delta(t-n)e^{-st}\,dt\right] = \sum_{n=-\infty}^{\infty} f(n)e^{-ns}$$

$$= \sum f(n)(e^{s})^{-n} = X(z)|z = e^{s} \qquad (3.7)$$

So the relation between Laplace-transform and Z-transform is

$$X(s) = X(z)|z = e^s \quad (3.8)$$

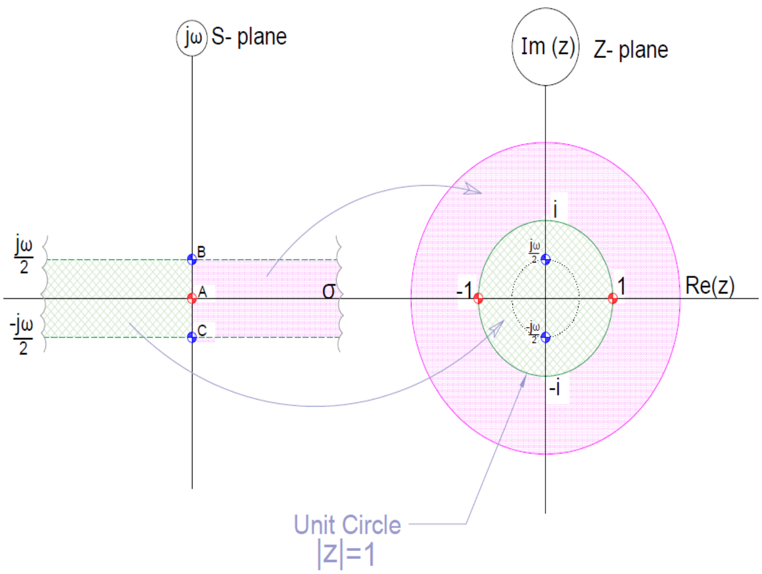
Or

$$X(z) = X(s)|s = \ln z \quad (3.9)$$

The most important correspondence between the s-plane and Z-plane are:

- The points on the $j\omega - axis$ in the s-plane mapped onto the unit circle in the Z-plane.
- The points in the right half on the s-plane mapped outside the unit circle in the Z-plane where the points in the left half mapped inside the unit circle.
- The lines$s = \sigma$ parallel to the the $j\omega$ -axis in the s-plane mapped into circle with radius $[z] = e^{\sigma}$ in z-plane where the lines$s = ju$ parallel to the σ-axis mapped into rays of thearg $z = u$ radians from $z = 0$.
- The origin of the s-plane mapped to the point $z = 1$in the Z-plane.

Mapping of different areas of s-plane onto the Z-plane is shown below.



Conclusion

For finding the inverse of Z-transform three methods are used: integration method is useful for finding a few values of an infinite complex sequence where the power series method is efficient when we find the inverse of finite-order integer power function with partial fraction method is suitable for finding the inverse of rational Z-transform. Z-transform is a transform for discrete data equivalent to Laplace transform for continuous data and it's a generalization of discrete Fourier transform. Also, we can conclude there is close relationship between Fourier, Laplace and Z-transform. One transform can be inherited from another by changing the format of the variable s or z

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